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ORIGINAL RESEARCH

The third and hyper-Zagreb coindices of some graph operations

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Abstract In this paper some basic mathematical properties for the third and hyper Zagreb coindices of graph operations containing the Cartesian product and composition will be explained.

Keywords Hyper-Zagreb coindices · Third Zagreb coindices · Graph operations

Mathematics Subject Classification 05C76 · 05C07

1 Introduction

Mathematical calculations are absolutely necessary to explore important concepts in chemistry. In mathematical chemistry, molecules are often modeled by graphs named "molecular graphs". A molecular graph is a simple graph in which vertices are the atoms and edges are bonds between them. By IUPAC terminology, a topological index is a numerical value for correlation of chemical structure with various physical properties, chemical reactivity or biological activity.

We introduced have a new pair of invariants, the third Zagreb coindex and the hyper Zagreb coindex. It is well known that many graphs of general and in particular of chemical, interests arise from simpler graphs via various graph operations. It is, hence, important to understand how certain invariants of such composite graphs are related to the corresponding invariants of their components. Graovać and Pisanski [8] were the first to consider the problem of computing topological indices of product graphs. In their paper, they computed an exact formula for the Wiener index of the Cartesian

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product of graphs. The results were generalized by a series of authors who computed unweighted and vertex-weighted Wiener (or Hosoya) polynomials for various classes of composite graphs [6,17,19], including the Cartesian product, composition, sum, disjunction and symmetric difference of two graphs.

Throughout this paper we consider only simple connected graphs, i.e. connected graphs without loops and multiple edges. The Wiener index is the first and most studied topological indices, both from theoretical point of view and applications. It is equal to the sum of distances between all pairs of vertices of the respective graph, see for details [4,5,22]. We encourage the reader to consult [2,9,16,23–25] for historical background, computational techniques and mathematical properties of Zagreb indices.

The Cartesian product $G \times H$ of graphs G and H has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \times H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$. The Wiener index of the Cartesian product graphs was studied in [8,17].

The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent with u_2) or ($u_1 = u_2$ and v_1 is adjacent with v_2), see [[11], p.185].

Then, Ashrafi et al. [1] computed the first and second Zagreb coindices of the Cartesian product, composition, sum, disjunction and symmetric difference of two graphs. Here we continue this line of research by exploring the behavior of the third and hyper Zagreb coindices under several important operations. The results are applied to several classes of molecular graphs such as nanotubes and nanotori. In recent years, there has been considerable interest in general problems of determining topological indices [14,15].

2 Definitions and preliminaries

All graphs in this paper are finite and simple. For terms and concepts not defined here we refer the reader to any of several standard monographs such as, e.g., [3,10,20,21].

Let G be a finite simple graph on n vertices and m edges. We denote the vertex and the edge set of G by $V(G)$ and $E(G)$, respectively. The *complement* of G, denoted by $\bar{G}$, is a simple graph on the same set of vertices $V(G)$ in which two vertices u and v are *adjacent*, i.e., connected by an edge uv, if and only if they are not adjacent in G. Hence, $uv \in E\left(\bar{G}\right) \Leftrightarrow uv \notin E(G)$. Obviously, $E(G) \cup E\left(\bar{G}\right) = E(K_n)$ and $\bar{m} = E\left(\bar{G}\right) = \binom{n}{2} - m$. The degree of a vertex u in G is denoted by $d(u)$; the degree of the same vertex in $\bar{G}$ is then given by $d_{\bar{G}}(u) = n - 1 - d_G(u)$. We will omit the subscript G when the graph is clear from the context. For a (molecular) graph G, the *first Zagreb index* $M_1(G)$ is equal to the sum of the squares of the degrees of the vertices, and the *second Zagreb index* $M_2(G)$ is equal to the sum of the products of the degrees of pairs of adjacent vertices. In fact,

$$M_1(G) = \sum_{u \in V(G)} d(u)^2, \quad M_2(G) = \sum_{uv \in E(G)} d(u)\, d(v).$$

Also, one can rewrite the first Zagreb index as:

$$M_1(G) = \sum_{uv \in E(G)} (d(u) + d(v)).$$

The *third Zagreb index* was first introduced by Fath-Tabar [7]. This index is defined as follows:

$$M_3(G) = \sum_{uv \in E(G)} |d(u) - d(v)|.$$

The *first and second Zagreb coindices* were first introduced by Ashrafi et al. [1]. They are defined as follows:

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} (d(u) + d(v)), \quad \overline{M_2}(G) = \sum_{uv \notin E(G)} d(u)d(v).$$

The *hyper-Zagreb index* was first introduced by [18]. This index is defined as follows:

$$HM(G) = \sum_{uv \in E(G)} (d(u) + d(v))^2.$$

3 Main results

All considered operations are binary. Hence, we will usually deal with two finite and simple graphs, G_1 and G_2. For a given graph G_i, its vertex and edge sets will be denoted by V_i and E_i, respectively, and their cardinalities by n_i and m_i, respectively, where $i = 1, 2$. The number of edges in $\bar{G}_i$ is denoted by $\overline{m}_i$. When more than two graphs can be combined using a given operation, the values of subscripts will vary accordingly. We begin with the following crucial related to distance properties of some graph operations.

Lemma 3.1 *Let G and H be two connected graphs. Then we have:*

(a)
$$\begin{aligned}
&|V(G \times H)| = |V(G[H])| = |V(G)|\,|V(H)|,\\
&|E(G \times H)| = |E(G)|\,|V(H)| + |V(G)|\,|E(H)|,\\
&|E(G + H)| = |E(G)| + |E(H)| + |V(G)|\,|V(H)|,\\
&|E(G[H])| = |E(G)|\,|V(H)|^2 + |E(H)|\,|V(G)|,
\end{aligned}$$

(b) $G \times H$ *is connected if and only if G and H are connected,*

(c) If (a, b) *is a vertex of* $G \times H$ *then* $deg_{G \times H}((a, b)) = deg_G(a) + deg_H(b)$,

(d) If (a, b) *is a vertex of* $G[H]$ *then* $deg_{G[H]}((a, b)) = |V(H)|\, deg_G(a) + deg_H(b)$,

(e) If (a, b) *is a vertex of* $G + H$ *then, we have:*

$$deg_{G+H}(a) \begin{cases} deg_G(a) + |V(H)| & if\ a \in V(G) \\ deg_H(a) + |V(G)| & if\ a \in V(H). \end{cases}$$

Proof The parts (a) and (b) are consequence of definitions and some famous results of the book of Imrich and Klavzar [11]. For the proof of (c-e) were refer to [12,13]. □

Proposition 3.2 *Let G be a simple graph, then* $\bar{M}_3(G) = M_3(\bar{G})$.

Proof

$$\begin{aligned} \bar{M}_3(G) = \sum_{uv\notin E(G)} |d_G(u) - d_G(v)| &= \sum_{uv\in E(\bar{G})} |d_G(u) - d_G(v)| \\ &= \sum_{uv\in E(\bar{G})} |n - 1 - d_G(u) - n + 1 + d_G(v)| \\ &= \sum_{uv\in E(\bar{G})} |(n-1-d_G(u)) - (n - 1 - d_G(v))| \\ &= \sum_{uv\in E(\bar{G})} \left|d_{\bar{G}}(u) - d_{\bar{G}}(v)\right| = M_3(\bar{G}). \end{aligned}$$

□

Proposition 3.3 *Let G be a simple graph, then* $\overline{M}_3(\bar{G}) = M_3(G)$.

Proof

$$\begin{aligned} \bar{M}_3(\bar{G}) &= \sum_{uv\notin E(\bar{G})} \left|d_{\bar{G}}(u) - d_{\bar{G}}(v)\right| \\ &= \sum_{uv\in E(G)} |(n-1) - d_G(u) - (n-1) + d_G(v)| \\ &= \sum_{uv\in E(G)} |d_G(u) - d_G(v)| = M_3(G). \end{aligned}$$

Its follows directly from the definition that third Zagreb coindex achieve its smallest possible value of zero on complete, empty and on cycle graphs. □

Proposition 3.4

$\overline{M}_3(K_n) = M_3(\overline{K}_n) = 0,$
$\overline{M}_3(\overline{K}_n) = M_3(K_n) = 0,$
$\overline{M}_3(C_n) = M_3(\bar{C}_n) = 0,$
$\overline{M}_3(\bar{C}_n) = M_3(C_n) = 0.$

Proposition 3.5 *Let G be a simple graph with n vertices and m edges. Then* $HM(\bar{G}) = 4(n-1)^2\,\overline{m} - 4(n-1)\,\overline{M}_1(G) + \overline{HM}(G)$.

Proof

$$\begin{aligned}
HM\left(\bar{G}\right) &= \sum_{uv\in E(\bar{G})} \left(d_{\bar{G}}(u) + d_{\bar{G}}(v)\right)^2 \\
&= \sum_{uv\in E(\bar{G})} \left(n-1-d_G(u)+n-1-d_G(v)\right)^2 \\
&= \sum_{uv\notin E(G)} \left(2(n-1)-\left(d_G(u)+d_G(v)\right)\right)^2 \\
&= 4(n-1)^2 \sum_{uv\notin E(G)} 1 - 4(n-1) \sum_{uv\notin E(G)} \left(d_G(u)+d_G(v)\right) \\
&\quad + \sum_{uv\notin E(G)} \left(d_G(u)+d_G(v)\right)^2 \\
&= 4(n-1)^2\,\overline{m} - 4(n-1)\,\overline{M}_1(G) + \overline{HM}(G).
\end{aligned}$$

□

Proposition 3.6 *Let G be a graph. Then,* $\overline{HM}(G) = HM\left(\bar{G}\right) - 4(n-1)\,M_1\left(\bar{G}\right) + 4(n-1)^2\,\overline{m}$.

Proof

$$\begin{aligned}
\overline{HM}(G) &= \sum_{uv\notin E(G)} \left(d_G(u)+d_G(v)\right)^2 \\
&= \sum_{uv\in E(\bar{G})} \left(-d_G(u)-d_G(v)\right)^2 \\
&= \sum_{uv\in E(\bar{G})} \left[\left(n-1-d_G(u)-(n-1)\right)+\left(n-1-d_G(v)-(n-1)\right)\right]^2 \\
&= \sum_{uv\in E(\bar{G})} \left[d_{\bar{G}}(u)+d_{\bar{G}}(v)-2(n-1)\right]^2 \\
&= \sum_{uv\in E(\bar{G})} \left(d_{\bar{G}}(u)+d_{\bar{G}}(v)\right)^2 - 4(n-1)\sum_{uv\in E(\bar{G})} \left(d_{\bar{G}}(u)+d_{\bar{G}}(v)\right) \\
&\quad + 4(n-1)^2 \sum_{uv\in E(\bar{G})} 1 \\
&= HM\left(\bar{G}\right) - 4(n-1)\,M_1\left(\bar{G}\right) + 4(n-1)^2\,\overline{m}.
\end{aligned}$$

□

Proposition 3.7 *Let G be a graph. Then,* $\overline{HM}\left(\bar{G}\right) = 4m(n-1)^2 - 4(n-1)\,M_1(G) + HM(G)$.

Proof

$$\overline{HM}\left(\bar{G}\right) = \sum_{uv \notin E(\bar{G})} \left(d_{\bar{G}}(u) + d_{\bar{G}}(v)\right)^2$$
$$= \sum_{uv \in E(G)} \left(2(n-1) - (d_G(u) + d_G(v))^2\right)$$
$$= 4(n-1)^2 \sum_{uv \in E(G)} 1 - \left(4(n-1) \sum_{uv \in E(G)} (d_G(u) + d_G(v))\right)$$
$$+ \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2$$
$$= 4m(n-1)^2 - 4(n-1) M_1(G) + HM(G).$$

The following results for complete, cycle, path, complete bipartite and hyper cube graph follow easily by direct calculations. □

Proposition 3.8 *(1)* $\overline{HM}(K_n) = 0$,

(2) $\overline{HM}(C_n) = 8n(n-3)$,

(3) $\overline{HM}(P_n) = 8n^2 - 38n + 46$,

(4) $\overline{HM}\left(K_{m,n}\right) = 2mn\left[n(m-1) + m(n-1)\right]$,

(5) $\overline{HM}(Q_k) = 2^{1+K}K^2\left(2^K - K - 1\right)$.

Here we consider the union operation of two graphs. A *union* $G_1 \cup G_2$ of the graphs G_1 and G_2 is the graph with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. Here we assume that V_1 and V_2 are disjoint.

Proposition 3.9 *Let G_1 and G_2 be two simple graphs. Then*

(1) $\overline{M}_3(G_1 \cup G_2) \leq \overline{M}_3(G_1) + \overline{M}_3(G_2) + 2(m_1 n_2 + m_2 n_1)$,
(2) $\overline{HM}(G_1 \cup G_2) = \overline{HM}(G_1) + \overline{HM}(G_2) + n_2 M_2(G_1) + n_1 M_2(G_2) + 8m_1 m_2$.

Proof The degree $d_{G_1 \cup G_2}(u)$ of a vertex u is equal to the degree of u in the component G_i that contains it. The third Zagreb coindex $G_1 \cup G_2$ is equal to the sum of the third Zagreb coindices of the components plus the contributions from the missing edges between the components. There are $n_1 n_2$ of missing edges and their contribution is given by:

$$\sum_{u\in V(G_1)}\left[\sum_{v\in V(G_2)} |d(u)-d(v)|\right] \le \sum_{u\in V(G_1)}\left[\sum_{v\in V(G_2)} (|d(u)|+|d(v)|)\right]$$
$$= \sum_{u\in V(G_1)}\left[\sum_{v\in V(G_2)} (d(u)+d(v))\right]$$
$$= \sum_{u\in V(G_1)} [n_2 d(u)+2m_2] = 2m_1n_2+2m_2n_1.$$

This gives us the first claim. The second claim follows by the same reasoning. Since the contribution of the missing edges between G_1 and G_2 is given by:

$$\sum_{u\in V(G_1)}\left[\sum_{v\in V(G_2)} (d(u)+d(v))^2\right]$$
$$= \sum_{u\in V(G_1)}\left[\sum_{v\in V(G_2)} d(u)^2+\sum_{v\in V(G_2)} d(v)^2+2\sum_{v\in V(G_2)} d(u)\,d(v)\right]$$
$$= \sum_{v\in V(G_2)}\sum_{u\in V(G_1)} d(u)^2+\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d(v)^2+2\sum_{u\in V(G_1)} d(u)\sum_{v\in V(G_2)} d(v)$$
$$= n_2M_2(G_1)+n_1M_2(G_2)+8m_1m_2.$$

The result of Proposition 3.9 follows by the identity $\sum_{u\in V(G)} d(u)=2m$.

Next is the operation of *sum* of two graphs. A sum G_1+G_2 of two graphs G_1 and G_2 is sometimes called a *join*, and is denoted by $G_1\nabla G_2$. We first consider the case when one of the components in a sum is single vertex. □

Proposition 3.10 *(1)* $\overline{M}_3(G+K_1)=\overline{M}_3(G)$,
(2) $\overline{HM}(G+K_1)=\overline{HM}(G)+4\overline{M}_1(G)+4\overline{m}$.

Proof The degree $d_{G+K_1}(u)=d_G(u)+1$. Hence, to prove the first claim we have:

$$\bar{M}_3(G+K_1)=\sum_{uv\notin E(G)} \left|d_{G+K_1}(u)-d_{G+K_1}(v)\right|$$
$$= \sum_{uv\notin E(G)} |(d_G(u)+1)-(d_G(v)+1)|$$
$$= \sum_{uv\notin E(G)} |d_G(u)-d_G(v)| = \bar{M}_3(G).$$

For the second claim we have:

$$\overline{HM}(G+K_1)=\sum_{uv\notin E(G)} \left(d_{G+K_1}(u)+d_{G+K_1}(v)\right)^2$$

$$= \sum_{uv \notin E(G)} (d_G(u) + d_G(v) + 2)^2$$

$$= \sum_{uv \notin E(G)} (d_G(u) + d_G(v))^2$$

$$+ 4 \sum_{uv \notin E(G)} (d_G(u) + d_G(v)) + 4 \sum_{uv \notin E(G)} 1$$

$$= \overline{HM}(G) + 4\overline{M}_1(G) + 4\overline{m}.$$

□

Proposition 3.11 *Let G_1 and G_2 be two simple graphs. Then*

(1) $\overline{M}_3(G_1 + G_2) = \overline{M}_3(G_1) + \overline{M}_3(G_2)$,

(2) $\overline{HM}(G_1 + G_2) = \overline{HM}(G_1) + \overline{HM}(G_2) + 4\left(n_2\overline{M}_1(G_1) + n_1\overline{M}_1(G_2)\right) + 4\left(n_2^2\overline{m}_1 + n_1^2\overline{m}_2\right)$.

Proof To prove the first claim, notice that $d_{G_1+G_2}(u) = d_{G_1}(u) + n_2$ and $d_{G_1+G_2}(v) = d_{G_2}(v) + n_1$ for $u \in V(G_1), v \in V(G_2)$. Since all possible edges between G_1 *and* G_2 are present in $G_1 + G_2$, there are no missing edges, and hence their contributions is zero. The remaining two contribution one from the edges missing in G_1 and the other from the edges missing in G_2, are given by

$$\sum_{e \notin E(G_1)} \left|(d_{G_1}(u)+n_2)-(d_{G_1}(v)+n_2)\right| = \sum_{e \notin E(G_1)} \left|d_{G_1}(u) - d_{G_1}(v)\right| = \overline{M}_3(G_1),$$

and similarly for the sum over the edges missing in G_2 we have,

$$\sum_{e \notin E(G_2)} \left|(d_{G_2}(u)+n_1)-(d_{G_2}(v)+n_1)\right| = \sum_{e \notin E(G_2)} \left|d_{G_2}(u) - d_{G_2}(v)\right| = \overline{M}_3(G_2).$$

First claim now follows by adding two contributions. To prove the second claim we have the same reasoning.

$$\sum_{e \notin E(G_1)} \left[\left(d_{G_1}(u) + n_2\right) + \left(d_{G_1}(v) + n_2\right)\right]^2$$

$$= \sum_{e \notin E(G_1)} \left(d_{G_1}(u)+d_{G_1}(v)\right)^2 + 4n_2 \sum_{e \notin E(G_1)} \left(d_{G_1}(u)+d_{G_1}(v)\right) + 4n_2^2 \sum_{e \notin E(G_1)} 1$$

$$= \overline{HM}(G_1) + 4n_2\overline{M}_1(G_1) + 4n_2^2\overline{m}_1.$$

Similarly,

$$\sum_{e \notin E(G_2)} \left[\left(d_{G_2}(u) + n_1\right) + \left(d_{G_2}(v) + n_1\right)\right]^2$$

$$= \sum_{e \notin E(G_2)} \left(d_{G_2}(u)+d_{G_2}(v)\right)^2 + 4n_1 \sum_{e \notin E(G_2)} \left(d_{G_2}(u)+d_{G_2}(v)\right) + 4n_1^2 \sum_{e \notin E(G_2)} 1$$
$$= \overline{HM}(G_2) + 4n_1\overline{M}_1(G_2) + 4n_1^2\overline{m}_2.$$

Hence,

$$\overline{HM}(G_1+G_2) = \overline{HM}(G_1) + \overline{HM}(G_2) + 4\left(n_2\overline{M}_1(G_1) + n_1\overline{M}_1(G_2)\right) + 4\left(n_2^2\overline{m}_1 + n_1^2\overline{m}_2\right).$$

The sum operation can be extended inductively to more than two graphs in an obvious way. Let $G_1, \ldots, G_k$ be graphs with vertex sets V_i and edge sets E_i of cardinality n_i and m_i, respectively. □

Corollary 3.12 *(1)* $\overline{M}_3\left(\sum_{i=1}^{K} G_k\right) = \sum_{i=1}^{k} \overline{M}_3(G_k)$

(2) $\overline{HM}\left(\sum_{i=1}^{k} G_k\right) = \sum_{i=1}^{k} \overline{HM}(G_k) + 2k\left(\sum_{\substack{i,j=1\\ i\neq j}}^{k} n_j\overline{M}_1(G_i)\right) + k^2\left(\sum_{\substack{i,j=1\\ i\neq j}}^{k} n_j^2\overline{m}_i\right).$

In both Propositions 3.13 and 3.15 the third and hyper Zagreb coindices of the Cartesian product and composition of two graphs G_1 and G_2 are investigated.

Proposition 3.13 *(1)* $\overline{M}_3(G_1 \times G_2) \leq 2(n_1m_2 + n_2m_1)(n_1n_2 - 1) - n_1M_3(G_2) - n_2M_3(G_1)$.
(2) $\overline{HM}(G_1 \times G_2) = 2(n_1n_2 - 1)(n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2) - n_1 HM(G_2) - n_2HM(G_1) - 8(m_1M_1(G_2) + m_2M_1(G_1) + 2m_1m_2)$.

Proof To prove the first formula we have the expression $M_3(G_1 \times G_2) = n_2M_3(G_1) + n_1M_3(G_2)$ from Theorem 8 of [7].

$$\bar{M}_3(G_1 \times G_2) = \sum_{uv \notin E(G_1 \times G_1)} \left|d_{G_1 \times G_2}(u_1, u_2) - d_{G_1 \times G_2}(v_1, v_2)\right|$$
$$\leq \sum_{uv \notin E(G_1 \times G_1)} \left|d_{G_1}(u_1) - d_{G_1}(v_1)\right| + \sum_{uv \notin E(G_1 \times G_1)} \left|d_{G_2}(u_2) - d_{G_2}(v_2)\right|$$
$$\leq 2(n_1m_2 + n_2m_1)(n_1n_2 - 1) - M_3(G_1 \times G_2),$$
$$= 2(n_1m_2 + n_2m_1)(n_1n_2 - 1) - n_2M_3(G_1) - n_1M_3(G_2).$$

The second formula follows from the expression $HM(G_1 \times G_2) = n_1HM(G_2) + n_2HM(G_1) + 8m_1M_1(G_2) + 8m_2M_1(G_1) + 16m_1m_2$ from Theorem 2 of [18].

As an application of the above results, we give the explicit formula for the hyper Zagreb coindex and the upper bound formula for the third Zagreb coindex of $P_r \times$

P_s, $P_r \times C_q$ and $C_p \times C_q$. The formula follow from proportion 3.13 by plugging in the expressions:

$$M_3(P_n) = 2, M_3(C_n) = 0, HM(P_n) = 16n - 30 \quad and \quad HM(C_n) = 16n.$$

□

Corollary 3.14 *(1)* $\bar{M}_3(P_r \times P_s) \le 2(2rs - (r+s))(rs-1) - 2(r+s)$,
(2) $\overline{HM}(P_r \times P_s) = 4(rs-1)(8rs - 7r - 7s + 4) - 2(16rs - 15r - 15s) - 16(5rs - 6r - 6s + 7)$,
(3) $\overline{M}_3(P_r \times C_q) \le 2q(2r-1)(rq-1) - 2q$,
(4) $\overline{HM}(P_r \times C_q) = 2q(2(rq-1)(8r-7) + 63q - 5r)$,
(5) $\overline{M}_3(C_p \times C_q) \le 4pq(pq-1)$,
(6) $\overline{HM}(C_p \times C_q) = 2pq(16pq - 67)$.

Proposition 3.15 *(1)* $\overline{M}_3(G_1[G_2]) \le 2(n_1n_2 - 1)\left(m_1n_2^2 + n_1m_2\right) - 8m_1m_2n_2 - n_1M_1(G_2) - n_2^3M_1(G_1)$,
(2) $\overline{HM}(G_1[G_2]) = 2(n_1n_2-1)\left(n_2^3M_1(G_1) + n_1M_1(G_2) + 8m_1m_2n_2\right) - n_2^4HM(G_1) - n_1HM(G_2) - 2m_1n_2M_1(G_2) - 8n_2m_2(m_1M_1(G_2) + 2n_2M_1(G_1))$.

Proof

$$\begin{aligned}
\bar{M}_3(G_1[G_2]) &= \sum_{uv \notin E(G_1[G_2])} \left|d_{G_1[G_2]}(u_1,u_2) - d_{G_1[G_2]}(v_1,v_2)\right| \\
&= \sum_{uv \notin E(G_1[G_2])} \left|n_2 d_{G_1}(u_1) + d_{G_2}(u_2) - n_2 d_{G_1}(v_1) - d_{G_2}(v_2)\right| \\
&\le \sum_{uv \notin E(G_1[G_2])} n_2 \left|d_{G_1}(u_1) - d_{G_1}(v_1)\right| \\
&\quad + \sum_{uv \notin E(G_1[G_2])} \left|d_{G_2}(u_2) - d_{G_2}(v_2)\right| = 2(n_1n_2 - 1) \\
&\quad \times \left(m_1 n_2^2 + n_1 m_2\right) - 8m_1m_2n_2 - n_1M_1(G_2) - n_2^3 M_1(G_1),
\end{aligned}$$

The second proof follows from the expression $HM(G_1[G_2]) = n_2^4HM(G_1) + n_1HM(G_2) + 2m_1n_2M_1(G_2) + 8n_2m_2(m_1M_1(G_2) + 2n_2M_1(G_1))$ from Theorem 3 of [18]. □

As an application we present formulas for the third and hyper Zagreb coindices of $P_n[k_2]$ and $C_n[k_2]$.

Corollary 3.16 *(1)* $\overline{M}_3\left(P_n[K_2]\right) \leq 4\left(5n^2 - 19n + 18\right)$,
(2) $\overline{HM}\left(P_n\left[K_2\right]\right) = 200n^2 - 912n + 1032$,
(3) $\overline{M}_3\left(C_n[K_2]\right) \leq 20n\left(n-3\right)$,
(4) $\overline{HM}\left(C_n[K_2]\right) = \quad 8n\left(25n - 82\right)$.

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